NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF COMMUTATORS OF THE GENERAL FRACTIONAL INTEGRAL OPERATORS ON WEIGHTED MORREY SPACES

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ABSTRACT. We prove that b is in $Lip_\beta(\beta)$ if and only if the commutator $[b, L^{-\alpha/2}]$ of the multiplication operator by b and the general fractional integral operator $L^{-\alpha/2}$ is bounded from the weighed Morrey space $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, where $0 < \beta < 1$, $0 < \alpha + \beta < n, 1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 \leq k < p/q$, $\omega^{q/p} \in A_1$ and $r_\omega > \frac{1-k}{p/q-k}$, and here r_ω denotes the critical index of ω for the reverse Hölder condition.

1. Introduction and main results

Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}} \tag{1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$. Since we assume only upper bound on heat kernel $p_t(x, y)$ and no regularity on its space variables, this property (1) is satisfied by a class of differential operator, see [1] for details.

For $0 < \alpha < n$, the general fractional integral $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\frac{\alpha}{2}} f(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-tL} f \frac{dt}{t^{-\alpha/2+1}}(x).$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α which plays important roles in many fields. Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined by

$$[b, L^{-\alpha/2}]f(x) = b(x)L^{-\alpha/2}f(x) - L^{-\alpha/2}(bf)(x).$$

For the special case of $L = -\Delta$, many results have been produced. Paluszyński [7] obtained that $b \in Lip_\beta(\mathbb{R}^n)$ if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $1 < p < r < \infty, 0 < \beta < 1$ and $1/p - 1/r = (\alpha + \beta)/n$ with $p < n/(\alpha + \beta)$. Shirai [9] proved that $b \in Lip_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$ for $1 < p < q < \infty$, $0 < \alpha$, $0 < \beta < 1$ and $0 < \alpha + \beta = (1/p - 1/q)(n - \lambda) < n$ or $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ for $1 < p < q < \infty$, $0 < \alpha$, $0 < \beta < 1$, $0 < \alpha + \beta = (1/p - 1/q) < n$, $0 < \lambda < n - (\alpha + \beta)p$ and $\mu/q = \lambda/p$. Wang [12] established some weighted boundedness of properties of commutator $[b, I_\alpha]$ on

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the weighted Morrey spaces $L^{p,k}$ under appropriated conditions on the weight ω, where the symbol b belongs to (weighted) Lipschitz spaces. The weighted Morrey space was first introduced by Komori and Shirai [5]. For the general case, Wang [13] proved that if $b \in Lip_\beta(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$, where $0 < \beta < 1,\ 0 < \alpha + \beta < n,\ 1 < p < n/(\alpha+\beta), 1/p - 1/q = (\alpha+\beta)/n$ and $\omega^q \in A_1$.

The purpose of this paper is to give necessary and sufficient conditions for boundedness of commutators of the general fractional integrals with $b \in Lip_\beta(\omega)$ (the weighted Lipschitz space). Our theorems are the following:

Theorem 1.1. *Let $0 < \beta < 1,\ 0 < \alpha+\beta < n, 1 < p < \frac{n}{\alpha+\beta},\ 1/q = 1/p - (\alpha+\beta)/n$, $0 \le k < \min\{p/q, p\beta/n\}$ and $\omega^q \in A_1$. Then we have*

(a) If $b \in Lip_\beta(\mathbb{R}^n)$, then $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$;

(b) If $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$, then $b \in Lip_\beta(\mathbb{R}^n)$.

Theorem 1.2. *Let $0 < \beta < 1,\ 0 < \alpha+\beta < n, 1 < p < \frac{n}{\alpha+\beta},\ 1/q = 1/p - (\alpha+\beta)/n$, $0 \le k < p/q,\ \omega^{q/p} \in A_1$ and $r_\omega > \frac{1-k}{p/q-k}$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then we have*

(a) If $b \in Lip_\beta(\omega)$, then $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$;

(b) If $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, then $b \in Lip_\beta(\omega)$.

Our results not only extend the results of [12] from $(-\triangle)$ to a general operator L, but also characterize the (weighted) Lipschitz spaces by means of the boundedness of $[b, L^{-\alpha/2}]$ on the weighted Morrey spaces, which extend the results of [12] and [13]. The basic tool is based on a modification of sharp maximal function $M_L^\sharp$ introduced by [6].

Throughout this paper all notation is standard or will be defined as needed. Denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $\omega(B)$, where $\omega(B) = \int_B \omega(x)dx$. For a measurable set E, denote by χ_E the characteristic function of E. For a real number p, $1 < p < \infty$, let p' be the dual of p such that $1/p + 1/p' = 1$. The letter C will be used for various constants, and may change from one occurrence to another.

2. Some preliminaries

A non-negative function ω defined on $\mathbb{R}^n$ is called weight if it is locally integral. A weight ω is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C.$$

for every ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) \le C \operatorname*{essinf}_{x\in B} \omega(x).$$

When $p = \infty, \omega \in A_\infty$, if there exist positive constants δ and C such that given a ball B and E is a measurable subset of B, then

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^\delta.$$

A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$ if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C.$$

From the definition of $A_{p,q}$, we can get that

$$\omega \in A_{p,q} \ if \ and \ only \ if \ \omega^q \in A_{1+q/p'}. \tag{2}$$

Since $\omega^{q/p} \in A_1$, then by (2), we have $\omega^{1/p} \in A_{p,q}$.

A weight function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B \omega(x)^r dx\right)^{\frac{1}{r}} \le C\frac{1}{|B|}\int_B \omega(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 \le p < \infty$, then there exists $r > 1$ such that $\omega \in RH_r$. It follows from Hölders inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for all $1 < s < r$. Moreover, if $\omega \in RH_r, r > 1$, then we have $\omega \in RH_{r+\epsilon}$ for some $\epsilon > 0$. We thus write $r_w = \sup\{r > 1 : \omega \in RH_r\}$ to denote the critical index of ω for the reverse Hölder condition. For more details on Muchenhoupt class $A_{p,q}$, we refer the reader to [3], [10] and [11].

Definition 2.1. ([5]) *Let $1 \le p < \infty$ and $0 \le k < 1$. Then for two weights μ and ν, the weighted Morrey space is defined by*

$$L^{p,k}(\mu,\nu) = \{f \in L^p_{loc}(\mu) : \|f\|_{L^{p,k}(\mu,\nu)} < \infty\},$$

where

$$\|f\|_{L^{p,k}(\mu,\nu)} = \sup_B \left(\frac{1}{\nu(B)^k}\int_B |f(x)|^p \mu(x)dx\right)^{\frac{1}{p}}.$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

If $\mu = \nu$, then we have the classical Morrey space $L^{p,k}(\mu)$ with measure μ. When $k = 0$, then $L^{p,k}(\mu,\nu) = L^p(\mu)$ is the Lebesgue space with measure μ.

Definition 2.2. ([2]) *Let $1 \le p < \infty$, $0 < \beta < 1$, and $\omega \in A_\infty$. A locally integral function b is said to be in $Lip^p_\beta(\omega)$ if*

$$\|b\|_{Lip^p_\beta(\omega)} = \sup_B \frac{1}{\omega(B)^{\beta/n}}\left(\frac{1}{\omega(B)}\int_B |b(x) - b_B|^p \omega(x)^{1-p} dx\right)^{\frac{1}{p}} \le C < \infty,$$

where $b_B = |B|^{-1}\int_B b(y)dy$ and the supremum is taken over all ball $B \subset R^n$. When $p = 1$, we denote $Lip^p_\beta(\omega)$ by $Lip_\beta(\omega)$.

Obviously, for the case $\omega = 1$, then the $Lip^p_\beta(\omega)$ space is the classical Lip^p_β space.

Remark 2.1. *Let $\omega \in A_1$, García-Cuerva [2] proved that the spaces $\|f\|_{Lip^p_\beta(\omega)}$ coincide, and the norm of $\|\cdot\|_{Lip^p_\beta(\omega)}$ are equivalent with respect to different values of provided that $1 \le p < \infty$.*

Given a locally integrable function f and β, $0 \leq \beta < n$, define the fractional maximal function by

$$M_{\beta,r}f(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-\beta r/n}}\int_B |f(y)|^r dy\right)^{\frac{1}{r}}, \quad r \geq 1,$$

when $0 < \beta < n$. If $\beta = 0$ and $r = 1$, then $M_{0,\,1}f = Mf$ denotes the usual Hardy-Littlewood maximal function.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega f(x) = \sup_{x\in B}\frac{1}{\omega(B)}\int_B |f(y)|dy.$$

The fractional weighted maximal operator $M_{\beta,r,\omega}$ is defined by

$$M_{\beta,r,\omega}f(x) = \sup_{x\in B}\left(\frac{1}{\omega(B)^{1-\beta r/n}}\int_B |f(y)|^r\omega(y)dy\right)^{\frac{1}{r}},$$

where $0 \leq \beta < n$ and $r \geq 1$. For any $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, the sharp maximal function $M_L^\sharp f$ associated the generalized approximations to the identity $\{e^{-tL},\ t > 0\}$ is given by Martell [6] as follows:

$$M_L^\sharp f(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y) - e^{-t_B L}f(y)|dy,$$

where $t_B = r_B^2$ and r_B is the radius of the ball B. For $0 < \delta < 1$, we introduce the $\delta-$sharp maximal operator $M_{L,\delta}^\sharp$ as

$$M_{L,\delta}^\sharp f = M_L^\sharp(|f|^\delta)^{1/\delta},$$

which is a modification of the sharp maximal operator $M^\sharp$ of Fefferman and Stein ([10]). Set $M_\delta f = M(|f|^\delta)^{1/\delta}$. Using the same methods as those of [10] and [8], we can get

Lemma 2.1. *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound* (1). *Let $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Suppose that $\omega \in A_\infty$, then for every $0 < \eta < 1$, there exists a real number $\gamma > 0$ independent of γ, f such that we have the following weighted version of the local good λ inequality, for $\eta > 0$, $A > 1$,*

$$\omega\{x \in \mathbb{R}^n : M_\delta f > A\lambda, M_{L,\delta}^\sharp f(x) \leq \gamma\lambda\} \leq \eta\omega\{x \in \mathbb{R}^n : M_\delta f(x) > \lambda\}.$$

where $A > 1$ is a fixed constant which depends only on n.

If $\mu, \nu \in A_\infty, 1 < p < \infty, 0 \leq k < 1$, then

$$\|f\|_{L^{p,k}(\mu,\nu)} \leq \|M_\delta f\|_{L^{p,k}(\mu,\nu)} \leq C\|M_{L,\delta}^\sharp f\|_{L^{p,k}(\mu,\nu)}. \tag{3}$$

In particular, when $\mu = \nu = \omega$ and $\omega \in A_\infty$, we have

$$\|f\|_{L^{p,k}(\omega)} \leq \|M_\delta f\|_{L^{p,k}(\omega)} \leq C\|M_{L,\delta}^\sharp f\|_{L^{p,k}(\omega)}. \tag{4}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

Lemma 3.1. ([1]) *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound* (1). *Then for $0 < \alpha < 1$, the difference operator $L^{-\frac{\alpha}{2}} - e^{-tL}L^{-\frac{\alpha}{2}}$ has an associated kernel $K_{\alpha,t}(x,y)$ which satisfies*

$$K_{\alpha,t}(x,y) \leq \frac{C}{|x-y|^{n-\alpha}}\frac{t}{|x-y|^2}.$$

Lemma 3.2. *([12]) Let $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-(\alpha+\beta)/n$ and $\omega\in A_1$. Then for every $0<k<p/q$ and $1<r<p$, we have*

$$\|M_{\alpha+\beta,r}f\|_{L^{q,kq/p}(\omega^q)}\le C\|f\|_{L^{p,q}(\omega^p,\omega^q)}.$$

Lemma 3.3. *([5]) Let $0<\beta<n$, $1<p<n/\beta$, $1/s=1/p-\beta/n$ and $\omega\in A_{p,s}$. Then for every $0<k<p/s$, we have*

$$\|M_{\beta,1}f\|_{L^{s,ks/p}(\omega^s)}\le C\|f\|_{L^{p,k}(\omega^p,\omega^s)}.$$

Lemma 3.4. *([12]) Let $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-\alpha/n$, $1/s=1/q-\beta/n$ and $\omega^q\in A_1$. Then for every $0<k<p/s$, we have*

$$\|M_{\beta,1}f\|_{L^{s,ks/p}(\omega^s)}\le C\|f\|_{L^{q,kq/p}(\omega^q,\omega^s)}.$$

Lemma 3.5. *Let $0<\alpha+\beta<n$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-\alpha/n$, $1/s=1/q-\beta/n$ and $\omega^q\in A_1$. Then for every $0<k<p\beta/n$, we have*

$$\|L^{-\alpha/2}f\|_{L^{q,kq/p}(\omega^q,\omega^s)}\le C\|f\|_{L^{p,k}(\omega^p,\omega^s)}.$$

Proof. As before, we know that $L^{-\alpha/2}f(x)\le CI_\alpha(|f|)(x)$ for all $x\in\mathbb{R}^n$. Together with the result (cf. [12]), that is,

$$\|I_\alpha f\|_{L^{q,kq/p}(\omega^q,\omega^s)}\le C\|f\|_{L^{p,k}(\omega^p,\omega^s)},$$

we can get the desired result. □

Remark 3.1. *Using the boundedness property of I_α, we also know $L^{-\alpha/2}$ is bounded from L^1 to weak $L^{n/(n-\alpha)}$. It is easy to check that Lemma 3.2-3.5 also hold when $k=0$.*

The following lemma plays an important role in the proof of Theorem 1.1.

Lemma 3.6. *Let $0<\delta<1$, $0<\alpha<n$, $0<\beta<1$ and $b\in Lip_\beta(\mathbb{R}^n)$. Then for all $r>1$ and for all $x\in\mathbb{R}^n$, we have*

$$\begin{aligned}&M^\sharp_{L,\delta}([b,L^{-\alpha/2}]f)(x)\\ \le\;& C\|b\|_{Lip_\beta(\mathbb{R}^n)}\Big(M_{\beta,1}(L^{-\alpha/2}f)(x)+M_{\alpha+\beta,r}f(x)+M_{\alpha+\beta,1}f(x)\Big).\end{aligned}$$

The same method of proof as that of Lemma 4.6 (see below), we omit the details.

Proof of Theorem 1.1. We first prove (a). We only prove Theorem 1.1 in the case $0<\alpha<1$. For the general case $0<\alpha<n$, the method is the same as that of [1]. We omit the details.

For $0<\alpha+\beta<n$ and $1<p<n/(\alpha+\beta)$, we can find a number r such that $1<r<p$. By Eq.(4) and Lemma 3.6, we obtain

$$\begin{aligned}&\|[b,\ L^{-\alpha/2}]f\|_{L^{q,kq/p}(\omega^q)}\\ \le\;& C\|M^\sharp_{L,\delta}([b,\ L^{-\alpha/2}]f)\|_{L^{q,kq/p}(\omega^q)}\\ \le\;& C\|b\|_{Lip_\beta(\omega)}\Big(\|M_{\beta,1}(L^{-\alpha/2}f)\|_{L^{q,kq/p}(\omega^q)}\\ +\;& \|M_{\alpha+\beta,r}f\|_{L^{q,kq/p}(\omega^q)}+\|M_{\alpha+\beta,1}f\|_{L^{q,kq/p}(\omega^q)}\Big).\end{aligned}$$

Let $1/q_1 = 1/p - \alpha/n$ and $1/q = 1/q_1 - \beta/n$. Since $\omega^q \in A_1$, then by Eq.(2), we have $\omega \in A_{p,q}$. Since $0 < k < \min\{p/q,\ p\beta/n\}$, by Lemmas 3.2–3.5, we yield that

$$\begin{aligned}
& \|[b,\ L^{-\alpha/2}]f\|_{L^{q,kq/p}(\omega^q)} \\
\leq\ & C\|b\|_{Lip_\beta(\mathbb{R}^n)}\left(\|L^{-\alpha/2}f\|_{L^{q_1,kq_1/p}(\omega^{q_1},\ \omega^q)} + \|f\|_{L^{p,k}(\omega^p,\ \omega^q)}\right) \\
\leq\ & C\|b\|_{Lip_\beta(\mathbb{R}^n)}\|f\|_{L^{p,k}(\omega^p,\ \omega^q)}.
\end{aligned}$$

Now we prove (b). Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α. Let $k = 0$ and weight $\omega \equiv 1$, then $L^{p,k}(\omega^p, \omega^q) = L^p$ and $L^{q,kq/p}(\omega^q, \omega) = L^q$. From [7], the (L^p, L^q) bounedness of $[b, I_\alpha]$ implies that $b \in Lip_\beta(\mathbb{R}^n)$.

Thus Theorem 1.1 is proved. □

4. PROOF OF THEOREM 1.2

We also need some Lemmas to prove Theorem 1.2.

Lemma 4.1. *([12]) Let $0 < \alpha + \beta < n, 1 < p < \frac{n}{\alpha+\beta}, 1/q = 1/p - \alpha/n, 1/s = 1/q - \beta/n$ and $\omega^{s/p} \in A_1$. Then if $0 < k < p/s$ and $r_\omega > \frac{1}{p/q-k}$, we have*

$$\|M_{\beta,1}f\|_{L^{s,ks/p}(\omega^{s/p},\omega)} \leq C\|f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}.$$

Lemma 4.2. *([12]) Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$ and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$ and $r_\omega > \frac{1-k}{p/q-k}$, we have*

$$\|M_{\alpha,1}f\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Lemma 4.3. *([12]) Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, $0 < k < p/q$, $\omega \in A_\infty$. For any $1 < r < p$, we have*

$$\|M_{\alpha,r,\omega}f\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Lemma 4.4. *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$ and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$ and $r_\omega > \frac{1-k}{p/q-k}$, we have*

$$\|L^{-\alpha/2}f\|_{L^{q,kq/p}(\omega^{q/p},\ \omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Proof. Since the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1), it is easy to check that $L^{-\alpha/2}f(x) \leq CI_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$. Using the boundedness property of I_α on weighted Morrey space (cf. [12]), we have

$$\|L^{-\alpha/2}f\|_{L^{q,kq/p}(\omega^{q/p},\ \omega)} \leq \|I_\alpha f\|_{L^{q,kq/p}(\omega^{q/p},\ \omega)} \leq C\|f\|_{L^{p,k}(\omega)},$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. □

Remark 4.1. *It is easy to check that the above lemmas also hold for $k = 0$.*

Lemma 4.5. *Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1), and let $b \in Lip_\beta(\omega)$, $\omega \in A_1$. Then, for every function $f \in L^p(\mathbb{R}^n)$, $p > 1$, $x \in \mathbb{R}^n$, and $1 < r < \infty$, we have*

$$\sup_{x\in B} \frac{1}{|B|}\int_B |e^{-t_BL}(b(y) - b_B)f(y)|dy \leq C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).$$

Proof. Fix $f \in L^p(\mathbb{R}^n), 1 < p < \infty$ and $x \in B$. Then

$$\begin{aligned}
& \frac{1}{|B|}\int_B |e^{-t_B L}((b(\cdot)-b_B)f)(y)|dy \\
\leq \quad & \frac{1}{|B|}\int_B \int_{\mathbb{R}^n} |p_{t_B}(y,z)||(b(z)-b_B)f(z)|dzdy \\
\leq \quad & \frac{1}{|B|}\int_B \int_{2B} |p_{t_B}(y,z)||(b(z)-b_B)f(z)|dzdy \\
+ \quad & \frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B} |p_{t_B}(y,z)||(b(z)-b_B)f(z)|dzdy \\
\doteq \quad & \mathcal{M}+\mathcal{N}.
\end{aligned}$$

It follows from $y \in B$ and $z \in 2B$ that

$$|p_{t_B}(y,z)| \leq C t_B^{-n/2} \leq C\frac{1}{|2B|}.$$

Thus, Hölder's inequality and Definition 2.2 lead to

$$\begin{aligned}
\mathcal{M} \quad \leq \quad & C\frac{1}{|2B|}\int_{2B}|(b(z)-b_B)f(z)|dz \\
\leq \quad & C\frac{1}{|2B|}\left(\int_{2B}\|b(z)-b_B|^{r'}\omega(z)^{1-r'}dz\right)^{\frac{1}{r'}}\left(\int_{2B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}} \\
\leq \quad & C\|b\|_{Lip_\beta(\omega)}\frac{1}{|2B|}\omega(2B)^{\frac{\beta}{n}+\frac{1}{r'}}\omega(2B)^{\frac{1}{r}}\left(\frac{1}{\omega(2B)}\int_{2B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}} \\
\leq \quad & C\|b\|_{Lip_\beta(\omega)}\frac{1}{|2B|}\omega(2B)^{\frac{\beta}{n}+1}\left(\frac{1}{\omega(2B)}\int_{2B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}} \\
\leq \quad & C\|b\|_{Lip_\beta(\omega)}\omega(x)\left(\frac{1}{\omega(2B)^{1-\frac{\beta r}{n}}}\int_{2B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}} \\
\leq \quad & C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).
\end{aligned}$$

Moreover, for any $y \in B$ and $z \in 2^{k+1}B\setminus 2^kB$, we have $|y-z| \geq 2^{k-1}r_B$ and $|p_{t_B}| \leq C\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}$.

$$\begin{aligned}
\mathcal{N} \quad = \quad & \frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B} |p_{t_B}(y,z)||(b(z)-b_B)f(z)|dzdy \\
\leq \quad & C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}|(b(z)-b_B)f(z)|dz \\
\leq \quad & C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}|(b(z)-b_{2^{k+1}B})f(z)|dz \\
+ \quad & C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}|(b_{2^{k+1}B}-b_{2B})f(z)|dz \\
\doteq \quad & \mathcal{N}_1+\mathcal{N}_2.
\end{aligned}$$

We will estimate the values of terms $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively.

Using Hölder's inequality and Remark 2.1, we have

$$\begin{aligned}
\mathcal{N}_1 &\leq C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\\
&\quad\times\left(\int_{2^{k+1}B}|b(z)-b_B|^{r'}\omega(z)^{1-r'}dz\right)^{\frac{1}{r'}}\left(\int_{2^{k+1}B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}}\\
&\leq C\sum_{k=1}^{\infty}2^{(k+1)n}e^{-c2^{2(k-1)}}\\
&\quad\times\|b\|_{Lip_\beta(\omega)}\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}\left(\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^r\omega(z)dz\right)^{\frac{1}{r}}\\
&\leq C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).
\end{aligned}$$

Since $\omega\in A_1$, by the Hölder inequality, we get

$$\begin{aligned}
\mathcal{N}_2 &\leq C\sum_{k=1}^{\infty}2^{(k+1)n}e^{-c2^{2(k-1)}}\frac{k}{|2^{k+1}B|^{1-\beta r/n}}\omega(x)\|b\|_{Lip_\beta(\omega)}\int_{2^{k+1}B}|f(z)|dz\\
&\leq C\sum_{k=1}^{\infty}k2^{(k+1)n}e^{-c2^{2(k-1)}}\omega(x)\|b\|_{Lip_\beta(\omega)}\left(\frac{1}{|2^{k+1}B|^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&= C\sum_{k=1}^{\infty}k2^{(k+1)n}e^{-c2^{2(k-1)}}\\
&\quad\times\omega(x)\|b\|_{Lip_\beta(\omega)}\left(\frac{\omega(2^{k+1}B)^{1-\beta r/n}}{|2^{k+1}B|^{1-\beta r/n}}\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^rdz\right)^{\frac{1}{r}}\\
&\leq C\sum_{k=1}^{\infty}k2^{(k+1)n}e^{-c2^{2(k-1)}}\omega(x)\|b\|_{Lip_\beta(\omega)}\left(\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^r\omega(x)dz\right)^{\frac{1}{r}}\\
&\leq C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).
\end{aligned}$$

Thus Lemma 4.5 is proved. □

Lemma 4.6. *Let $0<\alpha<1$, $\omega\in A_1$ and $b\in Lip_\beta(\omega)$. Then for all $r>1$ and for all $x\in\mathbb{R}^n$, we have*

$$\begin{aligned}
&M^{\sharp}_{L,\delta}([b,L^{-\alpha/2}]f)(x)\leq C\|b\|_{Lip_\beta(\omega)}\\
&\times\left(\omega(x)^{1+\frac{\beta}{n}}M_{\beta,1}(L^{-\alpha/2}f)(x)+\omega(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,r,\omega}f(x)+\omega(x)^{1+\frac{\beta}{n}}M_{\alpha+\beta,1}f(x)\right).
\end{aligned}$$

Proof. For any given $x\in\mathbb{R}^n$, fix a ball $B=B(x_0,r_B)$ which contains x. We decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$. Observe that

$$[b,L^{-\alpha/2}]f(x)=(b-b_B)L^{-\alpha/2}f-L^{-\alpha/2}(b-b_B)f_1-L^{-\alpha/2}(b-b_B)f_2$$

and

$$e^{-t_BL}([b,L^{-\alpha/2}]f)=e^{-t_BL}[(b-b_B)L^{-\alpha/2}f-L^{-\alpha/2}(b-b_B)f_1-L^{-\alpha/2}(b-b_B)f_2].$$

Then

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left|[b,L^{-\alpha/2}]f(y)-e^{-t_BL}[b,L^{-\alpha/2}]f(y)\right|^{\delta}dy\right)^{1/\delta}\\
\leq\quad & C\left(\frac{1}{|B|}\int_B\left|(b(y)-b_B)L^{-\alpha/2}f(y)dy\right|^{\delta}\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\left|L^{-\alpha/2}(b(y)-b_B)f_1)(y)|dy\right|^{\delta}\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\left|e^{-t_BL}((b(y)-b_B)L^{-\alpha/2}f)(y)|dy\right|^{\delta}\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\left|e^{-t_BL}L^{-\alpha/2}((b(y)-b_B)f_1(y))|dy\right|^{\delta}\right)^{1/\delta}\\
&+C\left(\frac{1}{|B|}\int_B\left|(L^{-\alpha/2}-e^{-t_BL}L^{-\alpha/2})((b(y)-b_B)f_2)(y)|dy\right|^{\delta}\right)^{1/\delta}\\
\doteq\quad & I+II+III+IV+V.
\end{aligned}$$

We are going to estimate each term, respectively. Fix $0<\delta<1$ and choose a real number τ such that $1<\tau<2$ and $\tau'\delta<1$. Since $\omega\in A_1$, then it follows from Hölder's inequality that

$$\begin{aligned}
I\ &\leq\ C\left(\frac{1}{|B|}\int_B|(b(y)-b_B)|^{\tau\delta}dy\right)^{\frac{1}{\tau\delta}}\left(\int_B\left|L^{-\alpha/2}f(y)\right|^{\tau'\delta}dy\right)^{\frac{1}{\tau'\delta}}\\
&\leq\ C\left(\frac{1}{|B|}\int_B|(b(y)-b_B)|\,dy\right)\left(\int_B\left|L^{-\alpha/2}f(y)\right|dy\right)\\
&\leq\ C\|b\|_{Lip_\beta(\omega)}\frac{1}{|B|}\omega(B)^{1+\beta/n}\left(\int_B\left|L^{-\alpha/2}f(y)\right|dy\right)\\
&\leq\ C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1+\beta/n}M_{\beta,1}(L^{-\alpha/2}f)(x).
\end{aligned}$$

For II, using Hölder's inequality and Kolmogorov's inequality(see[3], p.485), then we deduce that

$$\begin{aligned}
II\ &\leq\ C\frac{1}{|B|}\int_B|L^{-\alpha/2}(b(y)-b_B)f_1)(y)|dy\\
&\leq\ C\frac{1}{|B|}|B|^{\frac{\alpha}{n}}\|L^{-\alpha/2}(b(y)-b_{2B})f_1\|_{L^{\frac{n}{n-\alpha}},\infty}\\
&\leq\ C\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B(b(y)-b_{2B})f_1(y)dy\\
&\leq\ C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,r,\omega}f(x).
\end{aligned}$$

Using Hölder's inequality and Lemma 4.5, we obtain that

$$III\leq C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}(L^{-\alpha/2}f)(x).$$

For IV, using the estimate in II, we get

$$\begin{aligned} IV &\leq \frac{C}{|B|}\int_B\int_{2B}|p_{t_B}(y,z)\|b(z)-b_B\|f(z)|dzdy \\ &\leq \frac{C}{|2B|}\int_{2B}L^{-\alpha/2}((b(z)-b_B))f(z)|dz \\ &\leq C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,r,\omega}f(x). \end{aligned}$$

By virtue of Lemma 3.1, we have

$$\begin{aligned} V &\leq \frac{C}{|B|}\int_B\int_{(2B)^c}|K_{\alpha,t_B}(y,z)\|(b(z)-b_B)f(z)|dzdy \\ &\leq \frac{C}{|B|}\sum_{k=1}^{\infty}\int_{2^k r_B\leq|x_0-z|<2^{k+1}r_B}\frac{1}{|x_0-z|^{n-\alpha}}\frac{r_B^2}{|x_0-z|^2}|(b(z)-b_B)f(z)|dz \\ &\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{k+1}B}|(b(z)-b_B)f(z)|dz \\ &\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{k+1}B}|(b(z)-b_{2^{k+1}B})f(z)|dz \\ &\quad +C\sum_{k=1}^{\infty}2^{-2k}(b_{2^{k+1}B}-b_B)\frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}}\int_{2^{k+1}B}|f(z)|dz \\ &\doteq VI+VII. \end{aligned}$$

Making use of the same argument as that of II, we have

$$VI\leq C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f(x).$$

Note that $\omega\in A_1$,

$$|b_{2^{k+1}B}-b_{2B}|\leq Ck\,\omega(x)\|b\|_{Lip_\beta(\omega)}\omega(2^{k+1}B)^{\beta/n}.$$

So, the value of VII can be controlled by

$$C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1+\beta/n}M_{\alpha+\beta,1}f(x).$$

Combining the above estimates for I–V, we finish the proof of Lemma 4.6. □

Proof of Theorem 1.2. We first prove (a). As before, we only prove Theorem 1.2 in the case $0<\alpha<1$. For $0<\alpha+\beta<n$ and $1<p<n/(\alpha+\beta)$, we can find a number r such that $1<r<p$. By Lemma 4.6, we obtain

$$\begin{aligned} &\|[b,\ L^{-\alpha/2}]f\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)} \\ \leq\ & C\|M^\sharp_{L,\delta}([b,\ L^{-\alpha/2}]f)\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)} \\ \leq\ & C\|b\|_{Lip_\beta(\omega)}\Big(\|\omega(\cdot)^{1+\frac{\beta}{n}}M_{\beta,1}(L^{-\alpha/2}f)\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)} \\ +\ & \|\omega(\cdot)^{1-\frac{\alpha}{n}}M_{\alpha+\beta,r,\omega}f\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)} \\ +\ & \|\omega(\cdot)^{1+\frac{\beta}{n}}M_{\alpha+\beta,1}f\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)}\Big) \\ \leq\ & C\|b\|_{Lip_\beta(\omega)}\Big(\|M_{\beta,1}(L^{-\alpha/2}f)\|_{L^{q,kq/p}(\omega^{q/p},\ \omega)} \end{aligned}$$

$$+ \quad \|M_{\alpha+\beta,r,\omega}f\|_{L^{q,kq/p}(\omega)} + \|M_{\alpha+\beta,1}f\|_{L^{q,kq/p}(\omega^{q/p},\ \omega)}\Big).$$

Let $1/q_1 = 1/p - \alpha/n$ and $1/q = 1/q_1 - \beta/n$. Lemmas 4.1–4.4 yield that

$$\begin{aligned} & \|[b,\ L^{-\alpha/2}]f\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\ \omega)} \\ \leq\ & C\|b\|_{Lip_\beta(\omega)}\left(\|L^{-\alpha/2}f\|_{L^{q_1,kq_1/p}(\omega^{q_1/p},\ \omega)} + \|f\|_{L^{p,k}(\omega)}\right) \\ \leq\ & C\|b\|_{Lip_\beta(\omega)}\|f\|_{L^{p,k}(\omega)}. \end{aligned}$$

Now we prove (b). Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α. We use the same argument as Janson [4]. Choose $Z_0 \in \mathbb{R}^n$ so that $|Z_0| = 3$. For $x \in B(Z_0, 2)$, $|x|^{-\alpha+n}$ can be written as the absolutely convergent Fourier series, $|x|^{-\alpha+n} = \sum_{m\in Z_n} a_m e^{i<\nu_m,x>}$ with $\sum_m |a_m| < \infty$ since $|x|^{-\alpha+n} \in C^\infty(B(Z_0, 2))$. For any $x_0 \in \mathbb{R}^n$ and $\rho > 0$, let $B = B(x_0, \rho)$ and $B_{Z_0} = B(x_0 + Z_0\rho, \rho)$,

$$\begin{aligned} & \int_B |b(x) - b_{B_{Z_0}}|dx = \frac{1}{|B_{Z_0}|}\int_B \left|\int_{B_{Z_0}} (b(x) - b(y))dy\right| dx \\ =\ & \frac{1}{\rho^n}\int_B s(x)\left(\int_{B_{Z_0}} (b(x) - b(y))|x-y|^{-\alpha+n}|x-y|^{n-\alpha}dy\right)dx, \end{aligned}$$

where $s(x) = \overline{\operatorname{sgn}\left(\int_{B_{Z_0}}(b(x) - b(y))dy\right)}$. Fix $x \in B$ and $y \in B_{Z_0}$, then $(y-x)/\rho \in B_{Z_0,2}$, hence,

$$\begin{aligned} & \frac{\rho^{-\alpha+n}}{\rho^n}\int_B s(x)\left(\int_{B_{Z_0}} (b(x) - b(y))\,|x-y|^{-\alpha+n}\left(\frac{|x-y|}{\rho}\right)^{n-\alpha} dy\right)dx \\ =\ & \rho^{-\alpha}\sum_{m\in Z^n} a_m \int_B s(x)\left(\int_{B_{Z_0}} (b(x) - b(y))\,|x-y|^{n-\alpha}e^{i<\nu_m,y/\rho>}dy\right)e^{-i<\nu_m,x/\rho>}dx \\ \leq\ & \rho^{-\alpha}\left|\sum_{m\in Z^n}|a_m|\int_B s(x)[b, L^{-\alpha/2}]\left(\chi_{B_{Z_0}}e^{i<\nu_m,\cdot/\rho>}\right)\chi_B(x)e^{-i<\nu_m,x/\rho>}dx\right| \\ \leq\ & \rho^{-\alpha}\sum_{m\in Z^n}|a_m|\|[b, L^{-\frac{\alpha}{2}}](\chi_{B_{Z_0}}e^{i<\nu_m,\cdot/\rho>})\|_{L^{q,0}(\omega^{1-(1-\alpha/n)q},\omega)}\left(\int_B \omega(x)^{q'(\frac{1}{q'}-\frac{\alpha}{n})}dx\right)^{\frac{1}{q'}} \\ \leq\ & C\rho^{-\alpha}\sum_{m\in Z^n}|a_m|\|\chi_{B_{Z_0}}\|_{L^{p,0}(\omega)}\left(\int_B \omega(x)^{q'(1/q'-\alpha/n)}dx\right)^{\frac{1}{q'}} \\ \leq\ & C\omega(B)^{1/p+1/q'-\alpha/n} = C\omega(B)^{1+\beta/n}. \end{aligned}$$

This implies that $b \in Lip_\beta(\omega)$. Thus, (b) is proved. □

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